Lecture 2

Wednesday, 1 February 2023 11:34 AM

Recall:

- Fisher market equilibrium problem
- market clearing prices, allocation
- Eisenberg - Gale convex program

Assume: ① $\forall j \in A$, $\exists i \in B$ s.t. $u_{ij} > 0$
② $\forall i \in B$, $e_i \in \mathbb{Z}_{++}$
③ $\forall i,j$, $u_{ij} \in \mathbb{Z}_+$

Theorem: ① Equilibrium (prices & allocation) exist
② The set of equilibrium allocations (x_{ij}) is convex
③ Prices & buyer utilities at equilibrium are unique
④ There exist rational & polynomially bounded equilibrium prices & allocation.

Recall: by KKT condns., x_{ij}'s are optimal soln. to E-G convex program iff $\exists$ p_j's s.t.:

1. $x_{ij} \geq 0$, & $\forall j$, $\sum_i x_{ij} = 1$ (feasibility)
2. $p_j \geq 0$ $\forall j$ (dual feasibility)
3. $p_j > 0 \Rightarrow \sum_i x_{ij} = 1$
4. $\forall i,j$, $\frac{u_i(x)}{e_i} \geq \frac{u_{ij}}{p_j}$
5. $\forall i,j$. $x_{ij} > 0 \Rightarrow \frac{u_i(x)}{e_i} = \frac{u_{ij}}{p_j}$

Proof of Theorem: ① was done last time

③ Utilities are unique because the objective of the E-G program is strictly concave.
Prices are unique because: Suppose $\exists$ equilibria (x,p) & (y,q) w/ $p_j > q_j$. Note that $\forall i$, $u_i(x) = u_i(y)$.
Then consider i, i' s.t. $x_{ij} > 0$, $y_{i'j} > 0$.

$$p_j = \frac{u_{ij} \cdot e_i}{u_i(x)} = \frac{u_{ij} \cdot e_i}{u_i(y)} \leq q_j$$

Similarly $$q_j = \frac{u_{i'j}\, e_{i'}}{u_{i'}(x)} = \frac{u_{i'j}\, e_{i'}}{u_{i'}(y)} \leq p_j.$$ Hence $p_j = q_j$.

② Prove yourself

④ Let (x,p) be an arbitrary equilibrium. Let $K_i = \{j : x_{ij} > 0\}$.
Consider the LP, w/ variables x_{ij}, q_j:

$$\begin{array}{lrcl} \forall i, \forall j \notin S_i & x_{ij} & = & 0 \\ \forall i,j & x_{ij} & \geq & 0 \\ \forall j & \sum_i x_{ij} & = & 1 \\ \forall i,j. & \sum_{j'} u_{ij'} x_{ij'} & \geq & q_j \cdot e_i \cdot u_{ij} \\ \forall i, j \in K_i & \sum_{j'} u_{ij'} x_{ij'} & = & q_j \cdot e_i \cdot u_{ij} \end{array}$$

One can verify that (i) choosing x, $q_j := \frac{1}{p_j}$ satisfies these equations, and (ii) any solution x, q satisfies the KKT condns. (w/ $p_j = 1/q_j$ $\forall j$) ∎

The above gives us a way to compute prices, given eq. allocation. We want a way to do the opposite as well: given prices, check if the market clears.

Lemma: Given prices, we can check in poly-time if these are equilibrium prices. If so, we can obtain the equilibrium allocations as well, using a single max-flow computation.

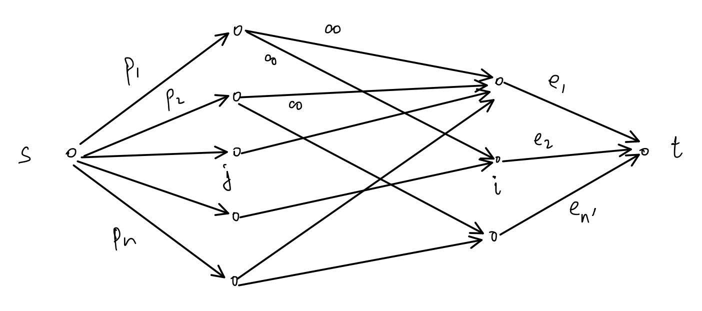
Proof: Given a Fisher market instance, & prices p_j, let
$\lambda_i = \max_j \frac{u_{ij}}{p_j}$. Note that, $\forall j$, $\frac{u_{ij}}{p_j} \leq \frac{u_i(x)}{e_i}$

& if $x_{ij} > 0$, then $\frac{u_{ij}}{p_j} = \frac{u_i(x)}{e_i}$. i.e.,

if $x_{ij} > 0$, then $\frac{u_{ij}}{p_j} = \max_{j'} \frac{u_{ij'}}{p_{j'}}$,

Thus, if x is an equilibrium allocation, then $x_{ij} > 0 \Rightarrow \frac{u_{ij}}{p_j} = \lambda_i$. Let $\Gamma(j) = \left\{ i : \frac{u_{ij}}{p_j} = \lambda_i \right\}$.

Consider the following flow network: $N(p)$



edge (j,i) exists if $i \in \Gamma(j)$, w/ capacity ∞

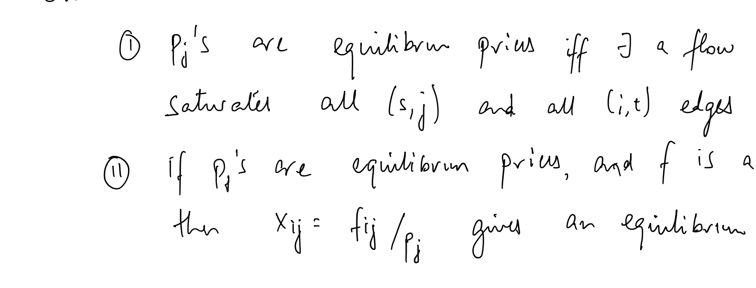
Can check that:

(i) p_j's are equilibrium prices iff $\exists$ a flow that saturates all (s,j) and all (i,t) edges

(ii) if p_j's are equilibrium prices, and f is a max-flow, then $x_{ij} = f_{ij}/p_j$ gives an equilibrium allocation. ∎

We will focus on max-flows & min s-t cuts in $N(p)$.

Consider any s-t cut $(\underbrace{s \cup A_1 \cup B_1}_{S=},\ \underbrace{t \cup (A \setminus A_1) \cup (B \setminus B_1)}_{T=})$



(i) if $\Gamma(A_1) \not\subseteq B_1$, $cap(S,T) = \infty$

(ii) if $\Gamma(A_1) \subseteq B_1$, $cap(S,T) = p(A \setminus A_1) + m(B_1)$

Our algorithm will start w/ prices $p_j = 1/\sum_{i,j'} u_{ij'}$ $\forall j$
Note that, at equilibrium, the prices must be at least these, by the KKT condns., & since e_i, u_{ij}'s are integers.
It will maintain that prices $\leq$ eq. prices.
In particular will maintain

Invariant: $S = s$, $T = A \cup B \cup t$ is a min-cut

We will use another characterization of this invariant:

Lemma: The invariant is satisfied iff:
$\forall A_1 \subseteq A$, $p(A_1) \leq m(\Gamma(A_1))$

(notation: $\forall A_1 \subseteq A$, $p(A_1) = \sum_{j \in A_1} p_j$
$\forall B_1 \subseteq B$, $m(B_1) = \sum_{i \in B_1} e_i$)

i.e., invariant is satisfied iff for any subset of goods, the total price is at most the total endowment of interested buyers.

Proof: Say $\forall A_1 \subseteq A$, $p(A_1) \leq m(\Gamma(A_1))$.
Consider any s-t cut $(\underbrace{s \cup A_1 \cup B_1}_{=S}, \underbrace{(A \setminus A_1) \cup (B \setminus B_1) \cup t}_{=T})$

Then if $cap(S,T) \neq \infty$, $\Gamma(A_1) \subseteq B_1$
Hence
$$\begin{aligned} cap(S,T) &= p(A \setminus A_1) + m(B_1) \\ &= p(A) - p(A_1) + m(B_1) \\ &\geq p(A) - p(A_1) + m(\Gamma(A_1)) \geq p(A) \\ &= cap(s, t \cup A \cup B) \end{aligned}$$

Say $(s, t \cup A \cup B)$ is a min-cut.
Then for any S,T cut, $S = s \cup A_1 \cup \Gamma(A_1)$
$cap(S,T) = p(A) - p(A_1) + m(\Gamma(A_1)) \geq p(A)$
$\Rightarrow p(A_1) \leq m(\Gamma(A_1))$ ∎

Given prices p, we will be interested in a particular kind of max-flow in $N(p)$, called a "balanced flow"

Fix prices p.
For a flow f, for buyer i, define $\gamma_i(N,f) = e_i - f(i,t)$.
This is the money left w/ buyer i.

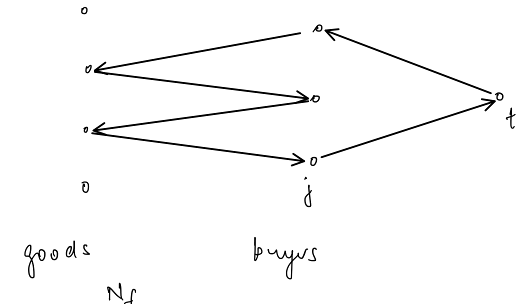
Defn: Flow f is balanced if it minimizes $\phi_N(f) := \sum_i \gamma_i(N,f)^2$

Note that: ① balanced flow is unique

Claim: If f is balanced, f must be a max-flow.
Proof: If not, then there is an augmenting path p from s to t in N_f, and $\delta > 0$ so that we can augment flow by δ along p. Let (i,t) be the last on p. Then $f+\delta$ reduces the excess for buyer i while having the other excesses unchanged, & hence f cannot be balanced.

Property 1: If i has lower excess/surplus than j, then there is no path from i to j in $N_f \setminus \{s,t\}$

Theorem: A max flow f is balanced iff it satisfies Property 1
Proof (sketch): Will prove contrapositive.
Suppose f violates Property 1.



Can show: $\exists$ $(j,t), (t,i)$ edges in N_f
then sending flow along this cycle increases $\gamma_i(N,f)$ & decreases $\gamma_j(N,f)$ by same amount, & hence reduces $\phi_N(f)$
$\Rightarrow f$ cannot be balanced flow

Suppose f is a max flow, but not balanced. Say g is a balanced flow.
Can show: Since g is also a max flow, $g-f$ consists of cycles. at least one cycle should include vertex t, else $\phi_N(f) = \phi_N(g)$
say the cycle includes the edges $(j,t), (t,i)$
if $\gamma_i(N,f) \leq \gamma_j(N,f)$, then shifting flow along this cycle will decrease (not increase) $\phi_N(f)$
Hence, $\gamma_i(N,f) > \gamma_j(N,f)$, & $N_f \setminus \{s,t\}$ contains an $i \to j$ path ∎

Finding a balanced flow in $N(p)$

We assume the invariant is satisfied, i.e., $(s, A \cup B \cup t)$ is a min-cut. Hence, $\forall A_1 \subseteq A$, $p(A_1) \leq m(\Gamma(A_1))$.

Step 1: Uniformly reduce capacities of (i,t) edges, until sum of these capacities $= p(A)$. If capacity of an edge reaches 0, stop reducing it, but keep reducing others.

Thus, choose δ s.t. $p(A) = \sum_{j \in B} \max\{0, e_i - \delta\}$

Let $e_i' = \max\{0, e_i - \delta\}$
& let $N'(p)$ be $N(p)$, but capacity of (i,t) edge is e_i'.

Step 2: Find a s-t min-cut (say $S = s \cup A_1 \cup B_1$, $T = t \cup (A \setminus A_1) \cup (B \setminus B_1)$)

Step 3: If $S = \{s\}$, return f, a max-flow in $N(p)$
Else, let N_1 be the graph $S \cup \{t\}$ (with additional (i,t) edges of capacity e_i, for $i \in B_1$)
N_2 be the graph $\{s\} \cup T$ (w/ addl. (s,j) edges of capacity p_j, for $j \notin A_1$)
Recursively, find balanced flows f_1 & f_2 in N_1 & N_2 respectively. Note that these are on disjoint edges.
Return the flow $f_1 \cup f_2$

We first show that f is a max-flow in $N(p)$

Lemma: The flow f is a max-flow in $N(p)$

Proof: We consider two cases, based on Step 3.
If $S = \{s\}$, i.e., $(s, A \cup B \cup t)$ is a min-cut in $N'(p)$.
Then f is a max-flow in $N'(p) \Rightarrow |f| = p(A) =$ min-cut in $N(p)$
Then f is a max-flow in $N(p)$ also.
Now say $S =$